

Boundary Dynamics Driven Entanglement

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We will show how it is possible to generate entangled states out of unentangled ones on a bipartite system by means of dynamical boundary conditions. The auxiliary system is defined by a symmetric but non-self-adjoint Hamiltonian and the space of self-adjoint extensions of the bipartite system is studied. It is shown that only a small set of them leads to separable dynamics and they are characterized. Various simple examples illustrating this phenomenon are discussed, in particular we will analyze the hybrid system consisting on a planar quantum rotor and a spin system under a wide class of boundary conditions.

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I. INTRODUCTION

There is an increasing interest in the physics associated to the “boundary” of a given physical system. As the boundary can be thought as an effective way of describing the interaction of the system with the external universe, its modeling could account for a number of significant physical phenomena. It is impossible to cover the range of physics associated to boundary structures. We will just mention here Casimir’s effect which is arguably one of the most conspicuous physical phenomena associated to presence of boundaries (see for instance [As06], [As08] and references therein for an extensive account of the role of boundary conditions and vacuum structures). We would like to add here the possibility of describing topology change as a boundary effect. This possibility was already considered in [Ba95] and further elaborated in relation with specific boundary condition in [As05], but it has gained new impetus because of Wilczek’s *et al* [Wi12] recent contribution to it.

In this paper we will explore how the manipulation of boundary conditions of composite systems allows to generate entangled states. More precisely, consider two systems A and B , and assume that the system B , which will be called the “bulk” or controlled system, is complete, i.e., its Hamiltonian H_B is an Hermitean (self-adjoint) operator on a Hilbert space \mathcal{H}_B and its evolution $U_t^B = \exp(itH_B)$ is unitary. However the system A , or “auxiliary”, is defined by a merely symmetric operator H_A on a Hilbert space \mathcal{H}_A . In other words the evolution “ $U_t^A = \exp(itH_A)$ ” will not be unitary until we have selected (if it exists) a self-adjoint extension of the operator H_A . It is worth to point it out here that such situation will actually happen whenever our system A is defined in a bounded domain Ω_A in \mathbb{R}^n with boundary $\partial\Omega_A$. In such case the Hilbert space \mathcal{H}_A will be the space $\mathcal{L}^2(\Omega)$ of square integrable complex-valued functions on

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Ω_A and the Hamiltonian operator will be:

$$H_A = -\frac{\hbar^2}{2m}\Delta_\eta + V, \quad (1)$$

with Δ_η the Laplace–Beltrami operator defined by some metric η on Ω_A , and V a potential function. Under such circumstances it can be shown that the self-adjoint extensions of H_A are determined by boundary conditions satisfied by the functions on the corresponding domain [As05].

The main observation which is relevant for the purposes of this paper, is that if we consider the bipartite system defined on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the family of self-adjoint extensions of the symmetric Hamiltonian $H = H_A \otimes I + I \otimes H_B$, is much larger than the family of self-adjoint extensions of the standalone symmetric Hamiltonian H_A . As it will be discussed along the paper, many of the self-adjoint extensions of the bipartite system are able to generate entangled states, i.e., they do not define separable dynamics, meaning by that they will not preserve separable states. Separable dynamics will be characterized and it will be shown that they correspond to the trivial extension to the tensor product (in a sense that will be described later on) of a self-adjoint extension of the system A . Such sources of entangled states will be called boundary generated entanglement.

It will be illustrated in simple examples how by choosing non-trivial tensor product extensions of a given self-adjoint extension of the system A , we obtain non-separable dynamics. Even more, we will show how by modifying the chosen self-adjoint extension, we can generate entangled states not only between the auxiliary system A and system B , but even in system B itself (as long as it is a composite system itself).

Such instances will be discussed first by using a toy example consisting of the free particle moving on the half-line as auxiliary system and a two-level system as a bulk system. In this particular instance it will be shown that the ground state of the half-line (actually its only eigenstate) becomes entangled with the eigenstates of the bulk system and how such entangled state can be driven by modifying the boundary conditions compatible with such scenario. Finally, we will discuss a “quantum compass”, i.e. a planar rotor possessing a spin 1/2 system sitting inside it. Now, two families of non-trivial boundary conditions for such system, extending in a nontrivial way quasiperiodic boundary conditions for the planar rotor [As83], will be considered and their spectral properties will be discussed.

II. BOUNDARY CONDITIONS AND SELF-ADJOINT EXTENSIONS

We will start by reviewing briefly the most salient aspects of the relation between self-adjoint extensions and boundary conditions by using the Laplace–Beltrami operator as an illuminating example.

Given a symmetric operator T on a Hilbert space \mathcal{H} , i.e., T has dense domain $\mathcal{D}_0 \subset \mathcal{H}$ and $T \subset T^\dagger$, we may use von Neumann’s theorem [Ne31] (see [Re75] for an exhaustive account of the theory) to describe all its self-adjoint extensions. Namely, we compute the deficiency spaces $\mathcal{N}_\pm = \ker(T^\dagger \mp i\mathbb{I}) = \text{Ran}(T \pm i\mathbb{I})^\perp$ and then there is a one-to-one correspondence between self-adjoint extensions of T and unitary operators $K: \mathcal{N}_+ \rightarrow \mathcal{N}_-$. To any such K we associate the operator T_K with domain:

$$\mathcal{D}_K = \mathcal{D}_0 \oplus (\mathbb{I} + K)\mathcal{N}_+ \quad (2)$$

and such that

$$T(\Phi_0 \oplus (\mathbb{I} + K)\xi_+) = T\Phi_0 \oplus i(\mathbb{I} - K)\xi_+, \quad \forall \Phi_0 \in \mathcal{D}_0, \xi_+ \in \mathcal{N}_+. \quad (3)$$

In many occasions the operator T , the Hamiltonian of the system for instance, is defined as a differential operator on a manifold Ω with boundary $\partial\Omega$. Let us consider in what follows, as an illustrative situation, the case of a free particle moving on a curved manifold Ω with Riemannian metric η . In such cases, the Hamiltonian describing the geodesic motion is the Laplace–Beltrami operator Δ_η (this is, we are in the situation of eq. (1) with $V \equiv 0$), that in local coordinates x^i , $i = 1, \dots, n$, $n = \dim \Omega$, takes the explicit form:

$$\Delta_\eta = \frac{1}{\sqrt{|\eta|}} \frac{\partial}{\partial x^i} \eta^{ij} \sqrt{|\eta|} \frac{\partial}{\partial x^j}, \quad (4)$$

with $|\eta| = \det(\eta_{ij})$. It is natural to select the domain of this operator as $C_c^\infty(\text{Int}(\Omega))$, i.e., as the set of complex-valued functions with compact support contained in the interior of Ω . This is a dense subspace of $\mathcal{H} = \mathcal{L}^2(\Omega)$, the space of square integrable functions with respect to the Riemannian volume defined by η . A simple integration by parts leads to

$$\langle \Phi, \Delta_\eta \Psi \rangle = \langle \Delta_\eta \Phi, \Psi \rangle \quad \forall \Phi, \Psi \in C_c^\infty(\text{Int}(\Omega)), \quad (5)$$

showing that the operator is symmetric. The minimal closed extension of the operator Δ_η is defined on the domain $\mathcal{D}_0 = \mathcal{H}_0^2(\Omega)$, i.e., the closure of $C_c^\infty(\text{Int}(\Omega))$ with respect to the Sobolev norm $\|\cdot\|_2$, which is just the Sobolev space of functions of order 2 vanishing at the boundary together with its normal derivative. The adjoint operator Δ_η^\dagger is the operator defined in the domain $\mathcal{D}_0^\dagger = \{\Phi \in \mathcal{L}^2(\Omega) \mid \Delta_\eta \Phi \in \mathcal{L}^2(\Omega)\}$. Such operator Δ_η^\dagger is actually the maximal extension of Δ_η and certainly $\Delta_\eta \subset \Delta_\eta^\dagger$.

A general result on operators commuting with conjugations shows the existence of self-adjoint extensions for Δ_η , then the existence of unitary operators $K: \mathcal{N}_+ \rightarrow \mathcal{N}_-$ and the applicability of Neumann’s theorem. eqs. (2), (3). Alternatively we may argue as follows (see for instance [As05] and references therein). Consider the restriction to the boundary $\partial\Omega$ of functions in \mathcal{D}_0^\dagger . Such restrictions will be denoted by $\varphi = \Phi|_{\partial\Omega}$. In the same way we define the normal derivative $\dot{\varphi} = \partial\Phi/\partial\nu|_{\partial\Omega}$ as

the outbound normal derivative along the boundary. We will consider that both $\varphi, \dot{\varphi}$ are in $\mathcal{L}^2(\partial\Omega)$ (even though this is not necessarily so and care must be taken when analyzing particular examples). Repeating now the computations leading to eq. (5) for elements $\Phi, \Psi \in \mathcal{D}_0^\dagger$ we will obtain:

$$\langle \Phi, \Delta_\eta \Psi \rangle - \langle \Delta_\eta \Phi, \Psi \rangle = \langle \varphi, \dot{\psi} \rangle - \langle \dot{\varphi}, \psi \rangle \quad (6)$$

where the inner product in the r.h.s. of the expression above is the one defined in $\mathcal{L}^2(\partial\Omega)$, i.e., $\langle \varphi, \psi \rangle = \int_{\partial\Omega} \bar{\varphi}(x) \psi(x) d\mu_{\partial\Omega}(x)$ with $\mu_{\partial\Omega}$ the measure associated to the Riemannian metric induced at the boundary $\partial\Omega$ by η . Clearly, self-adjoint extensions of Δ_η will be determined by maximal subspaces of functions Φ in \mathcal{D}_0^\dagger such that the bilinear form given by the r.h.s. of eq. (6) will vanish identically for the corresponding boundary values $\varphi, \dot{\varphi}$ of Φ .

Such maximally isotropic spaces W of boundary values can be easily characterized by computing their Cayley transform, i.e., we consider the linear isomorphism $C: \mathcal{L}^2(\partial\Omega) \oplus \mathcal{L}^2(\partial\Omega) \rightarrow \mathcal{L}^2(\partial\Omega) \oplus \mathcal{L}^2(\partial\Omega)$ defined by

$$C(\varphi, \dot{\varphi}) = \frac{1}{\sqrt{2}}(\varphi - i\dot{\varphi}, \varphi + i\dot{\varphi}).$$

Clearly C maps a maximally isotropic subspace W into the graph of a unitary operator $U: \mathcal{L}^2(\partial\Omega) \rightarrow \mathcal{L}^2(\partial\Omega)$. More explicitly $(\varphi, \dot{\varphi}) \in W$ iff there exists $U \in \mathcal{U}(\mathcal{L}^2(\partial\Omega))$ such that [As05]:

$$\varphi + i\dot{\varphi} = U(\varphi - i\dot{\varphi}). \quad (7)$$

In this sense the space of all self-adjoint extensions of the Laplace–Beltrami operator can be naturally identified with the unitary group of the Hilbert space of square integrable functions at the boundary of Ω and eq. (7) provides the explicit description of their domains. We will make extensive use of this characterization in what follows.

III. SELF-ADJOINT EXTENSIONS OF SYMMETRIC BIPARTITE SYSTEMS

Let us consider now the case of a bipartite system $A \times B$ such that one of its subsystems is described by a symmetric operator. In particular we will consider system A to be defined as in the previous section by the Laplace–Beltrami operator on a Riemannian manifold (Ω_A, η_A) , i.e., A is a free system on a manifold with boundary. System B will be defined by a self-adjoint operator H_B on a Hilbert space \mathcal{H}_B with dense domain $\text{dom}(H_B) = \mathcal{D}_B$. The Hilbert space \mathcal{H}_{AB} of pure states of the composite system is

$$\mathcal{H}_{AB} := \mathcal{H}_A \hat{\otimes} \mathcal{H}_B = \mathcal{L}^2(\Omega_A) \hat{\otimes} \mathcal{H}_B$$

that can be identified naturally with $\mathcal{L}^2(\Omega_A; \mathcal{H}_B)$, i.e. pure states will be considered as square integrable maps

$\Phi: \Omega_A \rightarrow \mathcal{H}_B$ with inner product

$$\langle \Phi, \Psi \rangle_{AB} = \int_{\Omega_A} \langle \Phi(x), \Psi(x) \rangle_{\mathcal{H}_B} d\mu_\eta(x). \quad (8)$$

In what follows we will use the latter identification when appropriate. The Hamiltonian operator will be $H = -\Delta_\eta \otimes \mathbb{I} + \mathbb{I} \otimes H_B$, that will act on states Φ as:

$$H\Phi = -\Delta_{\eta_A} \Phi + H_B \cdot \Phi, \quad (9)$$

with $(H_B \cdot \Phi)(x) = H_B(\Phi(x))$, $x \in \Omega_A$.

The natural symmetric domain \mathcal{D}_0 of the operator H is now $\mathcal{D}_0 = \mathcal{D}_{A0} \otimes \mathcal{D}_B$, where we are now borrowing the notation \mathcal{D}_{A0} from section II to denote the minimal closed extension of the Laplace–Beltrami operator defined in Ω_A . Again \mathcal{D}_0 can be identified in a natural way with $\overline{\mathcal{C}_0^\infty(\Omega_A; \mathcal{H}_B)}^{\|\cdot\|_2}$, where the completion is taken with respect to the Sobolev norm of order 2. Notice that we can not consider the completion $\hat{\otimes}$ in the definition of \mathcal{D}_0 because being \mathcal{D}_{A0} and \mathcal{D}_B dense, it would result $\mathcal{D}_{A0} \hat{\otimes} \mathcal{D}_B = \mathcal{H}_A \hat{\otimes} \mathcal{H}_B = \mathcal{H}_{AB}$ and the operator H is not bounded.

The maximal extension of this operator is given by $\mathcal{D}_{A0}^\dagger \otimes \mathcal{D}_B$ using the notation of section II again (notice that H_B is self-adjoint already). Computing the self-adjoint extensions of H is best done by using its boundary data structure (i.e., Green’s formula) like in the second part of section 2. In fact, integrating by parts we get the analogue of eq. (6):

$$\langle \Phi, -\Delta_{\eta_A} \Psi + H_B \cdot \Psi \rangle - \langle -\Delta_{\eta_A} \Phi + H_B \cdot \Phi, \Psi \rangle = \langle \varphi, \dot{\psi} \rangle - \langle \dot{\varphi}, \psi \rangle \quad (10)$$

where the inner product at the boundary appearing in the r.h.s. of the previous equation is given simply by:

$$\langle \varphi, \psi \rangle = \int_{\partial\Omega_A} \langle \varphi(x), \psi(x) \rangle_{\mathcal{H}_B} d\mu_{\partial\eta_A}(x) \quad (11)$$

and $\varphi, \dot{\varphi}$ are defined as before. The space of boundary data is now

$$\mathcal{L}^2(\partial\Omega_A; \mathcal{H}_B) \simeq \mathcal{L}^2(\partial\Omega_A) \hat{\otimes} \mathcal{H}_B.$$

Then, repeating the argument leading to eq. (7), we obtain that the space of self-adjoint extensions of T , i.e. the space of maximally isotropic closed subspaces of the bilinear boundary form defined by the r.h.s. of eq. (10), is parametrized by unitary operators $U \in \mathcal{U}(\mathcal{L}^2(\partial\Omega_A) \hat{\otimes} \mathcal{H}_B)$. Thus, given such an unitary U , the domain \mathcal{D}_U of the corresponding self-adjoint extension will consist of all functions $\Phi \in \mathcal{D}_{A0}^\dagger \otimes \mathcal{D}_B$ such that

$$\varphi + i\dot{\varphi} = U(\varphi - i\dot{\varphi}), \quad \varphi, \dot{\varphi} \in \mathcal{L}^2(\partial\Omega_A; \mathcal{H}_B). \quad (12)$$

The same result can be obtained certainly by using von Neuman’s Theorem. Because of the decomposition $\mathcal{D}_{A0}^\dagger = \mathcal{D}_{A0} \oplus (\mathcal{N}_{A+} \oplus \mathcal{N}_{A-})$ we will get:

$$\begin{aligned} \mathcal{D}_{A0}^\dagger \otimes \mathcal{D}_B &= [\mathcal{D}_{A0} \oplus (\mathcal{N}_{A+} \oplus \mathcal{N}_{A-})] \otimes \mathcal{D}_B \\ &= (\mathcal{D}_{A0} \otimes \mathcal{D}_B) \oplus [(\mathcal{N}_{A+} \oplus \mathcal{N}_{A-}) \otimes \mathcal{D}_B] \end{aligned} \quad (13)$$

Then the total deficiency space of the bipartite system will be:

$$\begin{aligned}\mathcal{N}_+ \oplus \mathcal{N}_- &= (\mathcal{N}_{A+} \oplus \mathcal{N}_{A-}) \hat{\otimes} \mathcal{D}_B \simeq (\mathcal{N}_{A+} \oplus \mathcal{N}_{A-}) \hat{\otimes} \mathcal{H}_B \\ &\simeq (\mathcal{N}_{A+} \otimes \mathcal{H}_B) \oplus (\mathcal{N}_{A-} \otimes \mathcal{H}_B).\end{aligned}\quad (14)$$

We can be more explicit by computing directly the deficiency spaces \mathcal{N}_\pm . If we choose an orthonormal basis ρ_n for \mathcal{H}_B then any vector $\Phi \in \mathcal{L}^2(\partial\Omega_A) \hat{\otimes} \mathcal{H}_B$ has a unique representation as :

$$\Phi = \sum_n \Phi_n \otimes \rho_n, \quad \Phi_n \in \mathcal{L}^2(\partial\Omega_A). \quad (15)$$

Hence if ρ_n are eigenvectors of H_B with corresponding eigenvalue λ_n (we will assume for simplicity that the spectrum of H_B is discrete) we will get:

$$\begin{aligned}(H^\dagger \mp iI)\Phi^\pm &= (-\Delta^\dagger \otimes \mathbb{I} + \mathbb{I} \otimes H_B) \sum_n \Phi_n^\pm \otimes \rho_n \mp iI\Phi^\pm = \\ &= \sum_n (-\Delta_{\eta_A}^\dagger \Phi_n^\pm + \lambda_n \Phi_n^\pm \mp i\Phi_n^\pm) \otimes \rho_n = 0\end{aligned}$$

which implies

$$-\Delta_{\eta_A}^\dagger \Phi_n^\pm \mp (i \mp \lambda_n) \Phi_n^\pm = 0. \quad (16)$$

Thus Φ_n^\pm must belong to the generalized deficiency spaces

$$\begin{aligned}\mathcal{N}_{A,z} &= \{\Phi^+ \mid H^\dagger \Phi^+ = z\Phi^+\}, \\ \mathcal{N}_{A,\bar{z}} &= \{\Phi^- \mid H^\dagger \Phi^- = \bar{z}\Phi^-\},\end{aligned}$$

with $z = (-\lambda_n + i)$. However all generalized deficiency spaces of the form $\mathcal{N}_{A,z}$ with either $\text{Im } z > 0$ or $\text{Im } z < 0$ are isomorphic (this is, $\dim \mathcal{N}_{A,z}$ is constant in both complex half-planes) and hence $\mathcal{N}_{A,\pm} = \mathcal{N}_{A,\pm i}$ is isomorphic to $\mathcal{N}_{A,\pm(i \mp \lambda_n)}$ (see for instance [Ak60]). Let us denote a choice for such isomorphism by $\alpha_n^\pm : \mathcal{N}_{A,\pm(i \mp \lambda_n)} \rightarrow \mathcal{N}_{A,\pm}$. Then we have shown that the deficiency spaces \mathcal{N}_\pm of the operator H consist of the vectors $\sum_n \Phi_n^\pm \otimes \rho_n$ with $\Phi_n^\pm \in \mathcal{N}_{A,\pm(i \mp \lambda_n)}$. The isomorphism $\alpha^\pm : \mathcal{N}_\pm \rightarrow \mathcal{N}_{A,\pm} \hat{\otimes} \mathcal{H}_B$ defined by

$$\alpha^\pm(\sum \Phi_n^\pm \otimes \rho_n) = \sum \alpha_n^\pm(\Phi_n^\pm) \otimes \rho_n \quad (17)$$

provides an explicit identification of \mathcal{N}_\pm with $\mathcal{N}_{A,\pm} \hat{\otimes} \mathcal{H}_B$ as indicated in eq. (14).

IV. SEPARABLE DYNAMICS AND SEPARABLE EXTENSIONS

It is clear that if we have two complete quantum systems A and B with Hilbert spaces of state vectors \mathcal{H}_A and \mathcal{H}_B , and Hamiltonian operators H_A and H_B respectively, then the bipartite system with Hilbert space vector states $\mathcal{H} = \mathcal{H}_A \hat{\otimes} \mathcal{H}_B$ and with total Hamiltonian $H = H_A \otimes \mathbb{I} + \mathbb{I} \otimes H_B$ induces a unitary flow

$$\begin{aligned}U_t &= e^{itH} = e^{it(H_A \otimes \mathbb{I} + \mathbb{I} \otimes H_B)} = e^{it(H_A \otimes \mathbb{I})} e^{it(\mathbb{I} \otimes H_B)} \\ &= e^{itH_A} \otimes e^{itH_B} = U_t^A \otimes U_t^B\end{aligned}$$

where U_t^A, U_t^B denote the individual unitary flows of the subsystems A and B . Then we may call a one-parameter family of unitary operators U_t on $\mathcal{H} = \mathcal{H}_A \hat{\otimes} \mathcal{H}_B$ separable if there exist two one-parameter families of unitary operators U_t^A and U_t^B on \mathcal{H}_A and \mathcal{H}_B respectively such that

$$U_t = U_t^A \otimes U_t^B. \quad (18)$$

Notice that U_t is separable if and only if $U_t\Phi$ is separable for any separable state $\Phi = \Phi_A \otimes \Phi_B$ for any t . In fact it is immediate to check that separable dynamics do not change the Schmidt index of a given state in $\mathcal{H}_A \hat{\otimes} \mathcal{H}_B$.

Now, if we are given a system H on $\mathcal{H}_A \otimes \mathcal{H}_B$ which is obtained by means of a self-adjoint extension of the product of a symmetric operator on \mathcal{H}_A and a self-adjoint operator on \mathcal{H}_B , can we characterize in which cases are we going to obtain separable dynamics? In other words, if $U \in \mathcal{U}(\mathcal{L}^2(\partial\Omega_A) \hat{\otimes} \mathcal{H}_B)$ is the unitary operator defining the self-adjoint extension eq. (12), when will it be separable?

We will solve first the spectral problem for the self-adjoint extension of H defined by the boundary condition $U = U_A \otimes \mathbb{I}$. We will assume for simplicity in what follows that the spectrum of H_B is discrete and nondegenerate. We denote the eigenvalues and eigenvectors of H_B by λ_k^B and ρ_k^B respectively, this is $H_B \rho_k^B = \lambda_k^B \rho_k^B$, $k = 1, 2, \dots$. An arbitrary function $\Phi \in \mathcal{L}^2(\Omega_A; \mathcal{H}_B)$ can be written uniquely as:

$$\Phi = \sum_{k=1}^{\infty} \Phi_k^A \otimes \rho_k^B,$$

hence

$$\varphi = \sum_{k=1}^{\infty} \varphi_k^A \otimes \rho_k^B, \quad \dot{\varphi} = \sum_{k=1}^{\infty} \dot{\varphi}_k^A \otimes \rho_k^B,$$

with $\varphi_k^A = \Phi_k^A|_{\partial\Omega_A}$ and $\dot{\varphi}_k^A = d\Phi_k^A/d\nu|_{\partial\Omega_A}$, $k = 1, 2, \dots$. If $U = \sum_{s=1}^N U_s^A \otimes U_s^B$ is a unitary operator acting on $\mathcal{L}^2(\partial\Omega_A) \hat{\otimes} \mathcal{H}_B$, we have $U\varphi = \sum_{k=1}^{\infty} \sum_{s=1}^N U_s^A \varphi_k^A \otimes U_s^B \rho_k^B$ and so on. In the particular instance of $U = U_A \otimes \mathbb{I}$ we get:

$$U\varphi = \sum_{k=1}^{\infty} U_A \varphi_k^A \otimes \rho_k^B$$

and similarly for $\dot{\varphi}$. Denoting as before by H_U the self-adjoint extension defined by U , the spectral problem $H_U\Phi = E\Phi$ becomes, after some trivial computations, the family of spectral problems:

$$H_A^\dagger \Phi_k^A + \lambda_k^B \Phi_k^A = E \Phi_k^A, \quad k = 1, 2, \dots \quad (19)$$

Notice however that the boundary conditions defined by U become the family of boundary conditions:

$$\varphi_k^A + i\dot{\varphi}_k^A = U_A(\varphi_k^A - i\dot{\varphi}_k^A), \quad k = 1, 2, \dots$$

Thus for each k we have to solve the problem:

$$\begin{cases} H_A^\dagger \Phi^A = (E - \lambda_k^B) \Phi^A \\ \varphi^A + i\dot{\varphi}^A = U_A(\varphi^A - i\dot{\varphi}^A). \end{cases} \quad (20)$$

Notice that if we denote by Ψ_l^A the eigenfunctions of the self-adjoint extension of the operator H_A defined by U_A , this is, with the boundary conditions given in eq. (20), we will have:

$$H_A^\dagger \Psi_l^A = \lambda_l^A \Psi_l^A, \quad \psi_l^A + i\dot{\psi}_l^A = U_A(\psi_l^A - i\dot{\psi}_l^A), \quad l = 1, 2, \dots$$

We will assume again, for simplicity, that the spectrum of the extension of H_A defined by U_A is discrete¹. We denote the corresponding operator by H_{U_A} instead of H_{A,U_A} . We finally conclude that the spectrum of H_U is given by:

$$E = \lambda_l^A + \lambda_k^B, \quad k, l = 1, 2, \dots,$$

with eigenvectors $\Psi_l^A \otimes \rho_k^B$. Now, if $\Phi \in \mathcal{L}^2(\Omega_A; \mathcal{H}_B)$, we will have:

$$\Phi = \sum_{k,l} c_{lk} \Psi_l^A \otimes \rho_k^B$$

with $c_{lk} = \langle \Psi_l^A \otimes \rho_k^B, \Phi \rangle$, and if Φ is separable, $\Phi = \Phi_A \otimes \Phi_B$, we will obtain:

$$c_{lk} = \langle \Psi_l^A, \Phi_A \rangle \langle \rho_k^B, \Phi_B \rangle = a_l b_k,$$

with $\Phi_A = \sum_l a_l \Psi_l^A$ and $\Phi_B = \sum_k b_k \rho_k^B$ respectively. Consequently,

$$\begin{aligned} e^{itH_U} \Phi_A \otimes \Phi_B &= \sum_{k,l} a_l b_k e^{itH_U} (\Psi_l^A \otimes \rho_k^B) \\ &= \sum_{k,l} a_l b_k e^{i(\lambda_l^A + \lambda_k^B)t} (\Psi_l^A \otimes \rho_k^B) \\ &= (e^{itH_{U_A}} \otimes e^{itH_B}) (\Phi_A \otimes \Phi_B) \end{aligned}$$

which shows that the self-adjoint extension defined by the unitary matrix $U = U_A \otimes \mathbb{I}$ is separable as it was easy to presume.

Let us discuss now boundary conditions of the form:

$$U = U_A \otimes U_B, \quad (21)$$

with $U_A \in \mathcal{U}(\mathcal{L}^2(\partial\Omega_A))$ and $U_B \in \mathcal{U}(\mathcal{H}_B)$, i.e., separable elements in the unitary group $\mathcal{U}(\mathcal{L}^2(\partial\Omega_A) \hat{\otimes} \mathcal{H}_B)$. We may even consider that the unitary U_B defines a symmetry of the quantum system H_B , this is $[H_B, U_B] = 0$. Now the answer is that, contrary to a simple guess, the

dynamics defined by U in eq. (21) is non-separable if $U_B \neq \mathbb{I}$.

The proof of this fact is as follows. Because U_B is a unitary operator, it can be diagonalized and the Hilbert space \mathcal{H}_B decomposed as $\mathcal{H}_B = \bigoplus_{s=1}^\infty W_s$, with W_s orthogonal U_B -invariant subspaces such that:

$$U_B \Phi_s^B = e^{i\nu_s} \Phi_s^B, \quad \forall \Phi_s^B \in W_s.$$

Now, because U_B commutes with H_B , H_B will leave the subspaces W_s invariant too, and we will denote by $H_{B,s}$ the restriction of H_B to W_s , $s = 1, 2, \dots$. On the other hand, we have:

$$\mathcal{H} = \mathcal{L}^2(\Omega_A; \mathcal{H}_B) = \mathcal{L}^2(\Omega_A; \bigoplus_{s=1}^\infty W_s) = \bigoplus_{s=1}^\infty \mathcal{L}^2(\Omega_A; W_s).$$

Moreover, the operator H_A^\dagger leaves invariant the subspaces $\mathcal{L}^2(\Omega_A; W_s)$ for all s . Hence, the spectral problem $H_U \Phi = E \Phi$ with boundary conditions defined by eq. (21) is equivalent to the solution of the family of spectral problems:

$$\begin{cases} (H_A^\dagger \otimes \mathbb{I} + \mathbb{I} \otimes H_B) \Phi_s = E_s \Phi_s \\ \varphi_s + i\dot{\varphi}_s = (U_A \otimes e^{i\nu_s})(\varphi_s - i\dot{\varphi}_s), \end{cases} \quad s = 1, 2, \dots, \quad (22)$$

because $\Phi_s \in \mathcal{L}^2(\Omega_A; W_s) \equiv \mathcal{L}^2(\Omega_A) \hat{\otimes} W_s$; $\varphi_s, \dot{\varphi}_s \in \mathcal{L}^2(\partial\Omega_A) \hat{\otimes} \mathcal{H}_B$, and $(U_A \otimes U_B)(\varphi_s - i\dot{\varphi}_s) = (U_A \otimes e^{i\nu_s})(\varphi_s - i\dot{\varphi}_s)$. But the boundary conditions in eq. (22) are the same as:

$$\varphi_s + i\dot{\varphi}_s = \tilde{U}_{A,s}(\varphi_s - i\dot{\varphi}_s), \quad s = 1, 2, \dots$$

with $\tilde{U}_{A,s} = e^{i\nu_s} U_A$, and in consequence the self-adjoint extension defined in $\mathcal{L}^2(\Omega_A; W_s)$ of the Hamiltonian $H_s = H_A \otimes \mathbb{I} + \mathbb{I} \otimes H_{B,s}$ by the boundary conditions $\tilde{U}_{A,s} \otimes \mathbb{I}$ is separable because of the argument in the previous paragraph. In consequence we have obtained that:

$$e^{itH_s} = e^{itH_{\tilde{U}_{A,s}}} \otimes e^{itH_{B,s}}.$$

Finally, if $\Phi = \Phi_A \otimes \Phi_B$ is a separable state in \mathcal{H} , we can write it as:

$$\Phi = \sum_{s=1}^\infty \sum_{i=1}^{N_s} \Phi_{s,i}^A \otimes \Phi_{s,i}^B$$

with $\Phi_{s,i}^B \in W_s$. Then:

$$\begin{aligned} e^{itH_U} \Phi &= \sum_{s=1}^\infty \sum_{i=1}^{N_s} e^{itH_U} \Phi_{s,i}^A \otimes \Phi_{s,i}^B \\ &= \sum_{s=1}^\infty \sum_{i=1}^{N_s} e^{itH_{\tilde{U}_{A,s}}} \Phi_{s,i}^A \otimes e^{itH_{B,s}} \Phi_{s,i}^B. \end{aligned} \quad (23)$$

But now the factor $e^{i\nu_s}$ is different for each s , hence the eigenvalues and eigenvectors of the spectral problem in

¹ This supposes no loss of generality for our purposes. In dimension 1 the spectrum of any self-adjoint extension of the Laplace operator is discrete [We80]. In general using the spectral theorem [Ak60], one can adapt this construction to the general case.

$\mathcal{L}^2(\Omega_A; W_s, s) = 1, 2, \dots, :$

$$\begin{cases} (H_A^\dagger \otimes \mathbb{I} + \mathbb{I} \otimes H_{B,s})\Phi_s = E_s \Phi_s \\ \varphi_s + i\dot{\varphi}_s = e^{i\nu_s} U_A(\varphi_s - i\dot{\varphi}_s), \quad s = 1, 2, \dots, \end{cases} \quad (24)$$

are different for each s . Therefore the extension $H_{\tilde{U}_{A,s}}$ is different for each s and we cannot factorize it out of the sum in the last term on the r.h.s. of eq. (23). Thus we conclude that if $U_B \neq \mathbb{I}$ the dynamics H_U is non-separable.

Actually we can prove the following theorem.

Theorem 1. *The dynamics H_U on the product Hilbert space $\mathcal{L}^2(\Omega_A) \hat{\otimes} \mathcal{H}_B$ defined by the unitary operator $U = U_A \otimes U_B \in \mathcal{U}(\mathcal{L}^2(\partial\Omega_A) \hat{\otimes} \mathcal{H}_B)$ is separable iff $U_B = \mathbb{I}$.*

Proof. Let assume that the dynamics defined by H_U is separable. Then:

$$e^{itH_U} = e^{it\tilde{H}_A} \otimes e^{it\tilde{H}_B},$$

where neither \tilde{H}_A nor \tilde{H}_B have to agree with H_{U_A} nor H_B respectively. However

$$H_U = \tilde{H}_A \otimes \mathbb{I} + \mathbb{I} \otimes \tilde{H}_B$$

with \tilde{H}_A and \tilde{H}_B self-adjoint operators. It is also clear that the one-parameter group of unitary operators:

$$V_t = e^{it\tilde{H}_B}$$

defines a symmetry group of H_U ,

$$[H_U, V_t] = 0, \quad \forall t.$$

Moreover the group V_t acts unitarily in the boundary space $\mathcal{L}^2(\partial\Omega_A; \mathcal{H}_B)$. Then it is easy to show that necessarily:

$$[U, I \otimes \tilde{H}_B] = 0,$$

hence $[U_B, \tilde{H}_B] = 0$. But now we have a self-adjoint extension defined by a unitary matrix of the form $U_A \otimes U_B$ with $[U_B, \tilde{H}_B] = 0$ as in the discussion preceding this theorem. Then, repeating the previous arguments we will obtain that the dynamics is non-separable unless $U_B = \mathbb{I}$. \square

V. A SIMPLE EXAMPLE: THE HALF-LINE/HALF-SPIN BIPARTITE SYSTEM

We will discuss now what is conceivably the simplest non-trivial example of a bipartite system of the kind considered in section 3. Let the auxiliary system A be a free particle moving on the half-line \mathbb{R}^+ ($\Omega_A = \mathbb{R}^+$, $\partial\Omega_A = \{0\}$) with Hilbert space $\mathcal{H}_A = \mathcal{L}^2(\mathbb{R}^+, dx)$ and the dynamics of such system is governed by the Hamiltonian $-\frac{1}{2} \frac{d^2}{dx^2}$. The bulk system B will be a 2-level system, for instance a spin 1/2 system whose Hilbert space is \mathbb{C}^2 .

The dynamics is given by an arbitrary 2×2 Hermitian matrix H_B . We will assume that $\sigma(H_B) = \{\lambda_1 > \lambda_2\}$ with eigenvectors ρ_a , $a = 1, 2$. The corresponding bipartite system $A \times B$ is defined in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2 = \mathcal{L}^2(\mathbb{R}^+) \hat{\otimes} \mathbb{C}^2 \simeq \mathcal{L}^2(\mathbb{R}^+; \mathbb{C}^2)$. Then state vectors $\Phi \in \mathcal{H}$ will be written as:

$$\Phi = \Phi_1 \otimes \rho_1 + \Phi_2 \otimes \rho_2 \simeq \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \end{bmatrix}, \quad \Phi_a(x) \in \mathcal{L}^2(\mathbb{R}^+), \quad (25)$$

where we use the orthonormal basis $\{\rho_1, \rho_2\}$ to write the component vectors.

As we showed before, the deficiency spaces are easy to compute and we get: $\mathcal{N}_\pm = \mathcal{N}_{A,\pm} \otimes \mathbb{C}^2 \simeq \mathbb{C}^2$, because, as it is easy to check, $\dim \mathcal{N}_{A,\pm} = 1$, i.e., $\mathcal{N}_{A,\pm} = \mathbb{C}$. However we will work directly with boundary values which will prove to be more efficient. Thus, given $\Phi \in \mathcal{H}$, the boundary values of Φ will live in $\mathcal{L}^2(\partial\mathbb{R}^+) \otimes \mathbb{C}^2$, in fact:

$$\varphi := \Phi|_{\partial\Omega_A} = \Phi_1|_{\partial\Omega_A} \otimes \rho_1 + \Phi_2|_{\partial\Omega_A} \otimes \rho_2 = \begin{bmatrix} \Phi_1(0) \\ \Phi_2(0) \end{bmatrix} =: \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$

and similarly

$$\begin{aligned} \dot{\varphi} &:= -\frac{\partial\Phi}{\partial x}|_{\partial\Omega_A} = \dot{\varphi}_1 \otimes \rho_1 + \dot{\varphi}_2 \otimes \rho_2 = \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}; \\ \dot{\varphi}_a &= \frac{\partial\Phi_a}{\partial x}|_{x=0}, \quad a = 1, 2. \end{aligned}$$

Finally:

$$\varphi_\pm = \varphi \pm i\dot{\varphi} = \begin{bmatrix} \varphi_1 \pm i\dot{\varphi}_1 \\ \varphi_2 \pm i\dot{\varphi}_2 \end{bmatrix}$$

and the self-adjoint extensions of $H = -\frac{d^2}{dx^2} \otimes \mathbb{I} + \mathbb{I} \otimes H_B$ are characterized by unitary operators $U \in \mathcal{U}(\mathcal{L}^2(\partial\Omega_A) \otimes \mathbb{C}^2) \simeq U(2)$.

Notice that in matricial form the operator H has the form:

$$-\frac{d^2}{dx^2} \otimes \mathbb{I} + \mathbb{I} \otimes H_B = \begin{bmatrix} -\frac{d^2}{dx^2} + \lambda_1 & 0 \\ 0 & -\frac{d^2}{dx^2} + \lambda_2 \end{bmatrix}. \quad (26)$$

We recall now that the boundary data space is given by $\mathcal{L}^2(\partial\mathbb{R}^+) \otimes \mathbb{C}^2$, hence according with the Thm. 1, separable dynamics will be given by unitary operators of the form $U = U_A \otimes \mathbb{I}$, where $U_A: \mathcal{L}^2(\partial\mathbb{R}^+) \rightarrow \mathcal{L}^2(\partial\mathbb{R}^+)$, hence $U_A = e^{i\alpha}$ is just multiplication by a phase. Incidentally we may recall that these are all the self-adjoint extensions of the system A in the half-line and they correspond to boundary conditions of the form:

$$\varphi_A + i\dot{\varphi}_A = e^{i\alpha}(\varphi_A - i\dot{\varphi}_A) \quad (27)$$

or equivalently

$$\begin{cases} \dot{\varphi}_A = \tan(\alpha/2)\varphi_A, & \alpha \neq \pi \\ \varphi_A = 0, & \alpha = \pi; \end{cases} \quad (28)$$

Now, because the space of self-adjoint extensions for the bipartite system is actually $U(2)$ as it was shown above, there are many self-adjoint extensions that will define non-separable dynamics.

We will consider the particular instance of self-adjoint extensions defined by the unitary matrices of the form:

$$U = U_A \otimes V, \quad U_A \in \mathcal{U}(\mathcal{L}^2(\partial\mathbb{R}^+)), V \neq \mathbb{I} \in U(2). \quad (29)$$

That despite of its form determines non-separable dynamical evolution. In fact, among this class and because U_A is just multiplication by a complex number of modulus 1, we can just consider as the simplest non-trivial example a matrix V of the form

$$V = \begin{bmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{i\alpha_2} \end{bmatrix}, \quad \text{with } e^{i\alpha_1} \neq e^{i\alpha_2}, \quad (30)$$

i.e. a matrix V belonging to the maximal compact tori inside $U(2)$. It is also noticeable that such V is the most general matrix commuting with H_B . Notice that if $\varphi = \varphi_1 \otimes \rho_1 + \varphi_2 \otimes \rho_2 \in \mathcal{L}^2(\partial\mathbb{R}^+) \otimes \mathcal{H}_B$, then

$$\begin{aligned} U\varphi &= (\mathbb{I} \otimes V)\varphi = \varphi_1 \otimes V\rho_1 + \varphi_2 \otimes V\rho_2 \\ &= \begin{bmatrix} V_{11}\varphi_1 + V_{12}\varphi_2 \\ V_{21}\varphi_1 + V_{22}\varphi_2 \end{bmatrix} = V \cdot \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = V \cdot \varphi. \end{aligned} \quad (31)$$

To compute the point spectrum of the self-adjoint operator H_U defined by the unitary operator $U = \mathbb{I} \otimes V$ is easy. Notice that eq. (12) becomes now:

$$\begin{bmatrix} \varphi_1 + i\dot{\varphi}_1 \\ \varphi_2 + i\dot{\varphi}_2 \end{bmatrix} = \begin{bmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{i\alpha_2} \end{bmatrix} \begin{bmatrix} \varphi_1 - i\dot{\varphi}_1 \\ \varphi_2 - i\dot{\varphi}_2 \end{bmatrix}, \quad (32)$$

this is, $\varphi_{a+} = e^{i\alpha_a} \varphi_{a-}$, $a = 1, 2$ or, if $\alpha_1, \alpha_2 \neq \pi$,

$$\dot{\varphi}_a = \tan(\alpha_a/2) \varphi_a, \quad a = 1, 2. \quad (33)$$

Then the eigenvalue problem $H_U \Phi = E\Phi$ becomes

$$\begin{aligned} H_U \Phi &= -\frac{d^2}{dx^2} \Phi_1 \otimes \rho_1 - \frac{d^2}{dx^2} \Phi_2 \otimes \rho_2 + \dots \\ &\quad \dots + \Phi_1 \otimes H_B \rho_1 + \Phi_2 \otimes H_B \rho_2 \\ &= \left(-\frac{d^2}{dx^2} + \lambda_1\right) \Phi_1 \otimes \rho_1 + \left(-\frac{d^2}{dx^2} + \lambda_2\right) \Phi_2 \otimes \rho_2 \\ &= E\Phi_1 \otimes \rho_1 + E\Phi_2 \otimes \rho_2. \end{aligned}$$

Then,

$$\left. \begin{aligned} -\frac{d^2}{dx^2} \Phi_a &= (E - \lambda_a) \Phi_a \\ \dot{\varphi}_a &= \tan(\alpha_a/2) \varphi_a \end{aligned} \right\} \quad a = 1, 2. \quad (34)$$

We may start solving:

$$-\frac{d^2 \Phi_1}{dx^2} = (E - \lambda_1) \Phi_1; \quad \varphi_1 = \tan(\alpha_1/2) \varphi_1. \quad (35)$$

Then we immediately see that if $\lambda_1 \leq E$, the solutions to this problem are not in $\mathcal{L}^2(\mathbb{R}^+)$, thus $\lambda_1 > E$ and

$\Phi_1(x) = C_1 e^{-\sqrt{\lambda_1 - E}x}$. Moreover $\dot{\varphi}_1 = -\frac{d\Phi_1}{dx}|_{x=0} = C_1 \sqrt{\lambda_1 - E}$, hence

$$\sqrt{\lambda_1 - E} = \tan(\alpha_1/2) \quad \text{or} \quad E = \lambda_1 - \tan^2(\alpha_1/2).$$

Notice that E is the unique discrete eigenvalue of the operator since the rest of the spectrum $(E, +\infty)$ is continuous. We can proceed similarly for the other component ($a = 2$) finding that $E = \lambda_2 - \tan^2(\alpha_2/2)$ if $E < \lambda_2$. In consequence, if $E < \lambda_2$ we obtain the compatibility condition²

$$\tan^2(\alpha_1/2) - \tan^2(\alpha_2/2) = \lambda_1 - \lambda_2 > 0, \quad (36)$$

that must be satisfied for the existence of an eigenvector with eigenvalue $\lambda_1 > \lambda_2 > E$. Figure 1 shows the space of self-adjoint extensions (α_1, α_2) with nondegenerate ground state E for various values of the spectral gap $\sigma = \lambda_1 - \lambda_2$ of the bulk system.

If $\lambda_2 \leq E < \lambda_1$, E is an eigenvalue again, but this time the eigenvector is going to have only the $a = 1$ component. We want to stress that the compatibility condition eq. (36) is only necessary for the existence of discrete spectrum. If it is not satisfied, then the problem has no discrete spectrum. However it still provides a well defined self-adjoint problem.

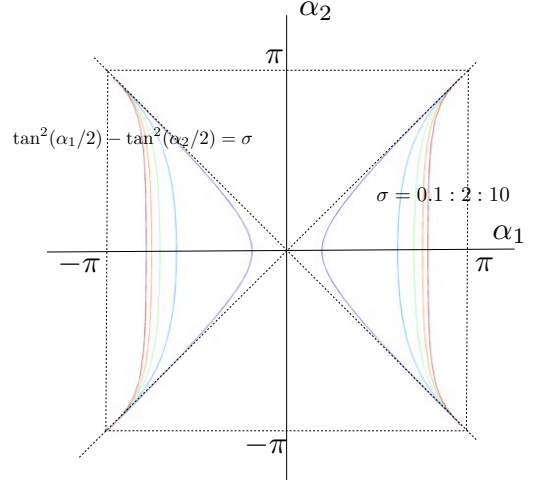


FIG. 1. Phase space of self-adjoint extensions of the half-line/half-spin system as function of the spectral gap σ possessing a unique ground state.

The curves defined by eq. (36) determined by the values of σ provide families of non-separable self-adjoint extensions of H compatible with the structure of H_B . Suppose now that we select as initial state the eigenstate corresponding to the extension defined by $\alpha_1 =$

² Remember the assumption $\lambda_1 > \lambda_2$ on the spectrum of the operator H_B .

$\arctan \sqrt{\sigma}$, $\alpha_2 = 0$, this is $\Phi_0 = e^{-\sqrt{\sigma}/2x} \otimes \rho_1$ and consider the one-parameter family of self-adjoint extensions defined by the unitary matrix:

$$U_s = \begin{bmatrix} e^{2is} & 0 \\ 0 & e^{2is'} \end{bmatrix} \quad (37)$$

with s, s' such that $\tan^2 s - \tan^2 s' = \sigma$, this is $s' = \arctan \sqrt{\tan^2 s - \sigma}$. Then if we proceed to modify the self-adjoint extension adiabatically³, the eigenstate Φ_0 will change with s but it will remain close to the (unique) eigenstate of the self-adjoint extension H_{U_s} , its ground state, that will be given by:

$$\Phi_s = C_1 e^{-(\tan s)x} \otimes \rho_1 + C_2 e^{-(\tan s')x} \otimes \rho_2, \quad 0 < s < \pi/2. \quad (38)$$

such state Φ_s is generically an entangled state in $\mathcal{H}_A \hat{\otimes} \mathcal{H}_B$. Notice that the phase diagram of the self-adjoint extensions constructed in this way is periodic and looks as shown in Figure 2 with black dots corresponding to separable states of the form either $e^{-\sqrt{\xi}x} \otimes \rho_1$ or $e^{-\sqrt{\xi}x} \otimes \rho_2$.

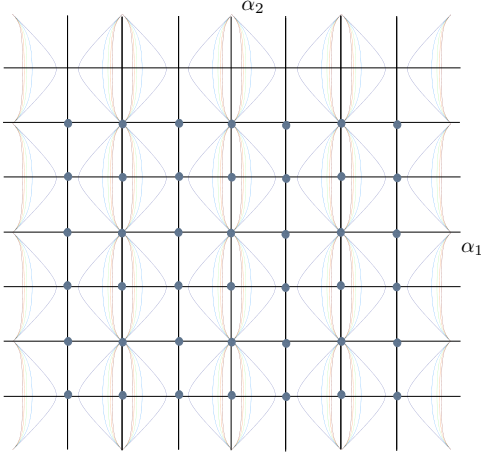


FIG. 2. Curves of self-adjoint extensions in the Abelian torus $\mathbb{T}^2 \subset U(2)$ with a single point spectrum.

The half-line/multipartite spin 1/2 system

We can elaborate the previous example again by considering a system B that is already a composite system, i.e. $\mathcal{H}_B = \mathcal{H}_{B_1} \hat{\otimes} \mathcal{H}_{B_2}$ with $\dim \mathcal{H}_{B_\alpha} = n_\alpha$, $\alpha = 1, 2$. The self-adjoint operators H_{B_α} , $\alpha = 1, 2$ have eigenvalues $\lambda_{k_\alpha}^{(\alpha)}$, $k_\alpha = 1, \dots, n_\alpha$ and a basis of eigenvectors of the operator $H_{B_1} \otimes \mathbb{I} + \mathbb{I} \otimes H_{B_2}$ given by

$$\rho_{k_1, k_2} = \rho_{k_1}^{(1)} \otimes \rho_{k_2}^{(2)}, \quad (39)$$

where $\rho_{k_\alpha}^{(\alpha)}$ are eigenvectors with eigenvalues $\lambda_{k_\alpha}^{(\alpha)}$. The eigenvalue corresponding to the eigenvector ρ_{k_1, k_2} is just $\lambda_{k_1}^{(1)} + \lambda_{k_2}^{(2)}$. Now we compute the system $A \times B$, to get

$$\mathcal{H} = \mathcal{H}_A \hat{\otimes} \mathcal{H}_B \simeq \mathcal{L}^2(\mathbb{R}^+; \mathcal{H}_{B_1} \hat{\otimes} \mathcal{H}_{B_2}) \quad (40)$$

and we expand $\Phi \in \mathcal{H}$ as :

$$\Phi = \sum_{1 \leq k_\alpha \leq n_\alpha} \Phi_{k_1, k_2} \otimes \rho_{k_1, k_2}. \quad (41)$$

In the same way $\varphi = \sum_{1 \leq k_\alpha \leq n_\alpha} \varphi_{k_1, k_2} \otimes \rho_{k_1, k_2}$ and $\dot{\varphi} = \sum_{1 \leq k_\alpha \leq n_\alpha} \dot{\varphi}_{k_1, k_2} \otimes \rho_{k_1, k_2}$ with $\varphi_{k_1, k_2} = \Phi(0)_{k_1, k_2}$ and $\dot{\varphi}_{k_1, k_2} = -\frac{d\Phi_{k_1, k_2}}{dx}|_{x=0}$. Finally, we notice that the space of self-adjoint extensions of the composite symmetric operator H is given by $\mathcal{U}(\mathcal{L}^2(\partial\mathbb{R}^+ \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}))$, i.e.

$$\mathcal{M}_{AB} = U(n_1 \cdot n_2) = U(N), \quad N = n_1 \cdot n_2, \quad n_\alpha = \dim \mathcal{H}_{B_\alpha} \quad (42)$$

Separable dynamics correspond to $U = U_A \times \mathbb{I}$, $U_A = e^{i\delta}$. Hence, let us choose boundary conditions leading to unseparable dynamics in the composite system $A \times B$ and in B itself.

We consider for instance $U = \mathbb{I} \times V$ with $V \in \mathcal{U}(\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2})$. Again, choosing a simplifying hypothesis, and considering the ordered spectrum of the Hamiltonian H_B , i.e. $\Lambda_1 = \max\{\lambda_{k_1}^{(1)} + \lambda_{k_2}^{(2)}\} = \lambda_{s_1}^{(1)} + \lambda_{s_2}^{(2)} \geq \Lambda_2 = \lambda_{r_1}^{(1)} + \lambda_{r_2}^{(2)} \geq \dots \geq \Lambda_N = \min\{\lambda_{k_1}^{(1)} + \lambda_{k_2}^{(2)}\}$ we denote the corresponding eigenvectors as Π_1, \dots, Π_N , then $H_B \Pi_l = \Lambda_l \Pi_l$. We choose now the matrix V to be diagonal in this basis,

$$V = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, \dots, e^{i\alpha_N})$$

and repeating the computations performed in the previous example we will get that the point spectrum of the operator H is given by $E = \Lambda_l - \tan^2 \alpha_l$, $l = 1, \dots, N$ which imposes $N - 1$ conditions on the parameters α_l of the form

$$\Lambda_l - \tan^2 \alpha_l = \Lambda_{l+1} - \tan^2 \alpha_{l+1}, \quad l = 1, \dots, N-1. \quad (43)$$

The previous equations (43) define a curve in the N -dimensional maximal compact abelian subgroup of $U(N)$ similar to those exhibited in Fig. 2. Again, a similar analysis as in the example of a single spin 1/2 system allows to conclude that an adiabatic deformation of the system along this curve will take a separable state, for instance $\Phi_{11} \otimes \rho_{11} = \Phi_{11} \otimes \rho_1^{(1)} \otimes \rho_1^{(2)}$ into (maximally) non-separable states.

VI. THE QUANTUM PLANAR ROTOR-SPIN SYSTEM

We will consider as a final example the interesting case of an hybrid system that captures some properties of electron-nucleus systems described recently (see [Sa12]).

³ If necessary we may reparametrize $s = s(t)$ where t is physical time, in such a way that $0 < ds/dt \ll 1$.

System A will be now a particle moving in the interval $\Omega_A = [0, 1]$ with measure dx , i.e. $\mathcal{H}_A = \mathcal{L}^2([0, 1], dx)$. Unlike in the previous case, now the boundary of system A has two points and therefore the self-adjoint extensions of system A alone is going to be parametrized by matrices in $U(2)$. Actually, we are going to consider a planar rotor with quasi-periodic boundary conditions [As83], i.e., the previous system with self-adjoint extensions determined by the unitary matrix

$$U_{A,\delta} = \begin{bmatrix} 0 & e^{i\delta} \\ e^{-i\delta} & 0 \end{bmatrix} \in \mathcal{U}(\mathcal{L}^2(\partial[0, 1])) , \quad (44)$$

that corresponds to boundary conditions $\Phi(0) = e^{i\delta}\Phi(1)$ and $\Phi'(0) = e^{i\delta}\Phi'(1)$. Now we will consider as bulk system B a two-level system, for instance a spin 1/2 system, with dynamics given by $H_B = \mu\sigma_z$, where σ_z is the diagonal Pauli matrix

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and μ is a constant that accounts for both, the coupling constant of the magnetic field with the spin 1/2 system and the strength of the magnetic field. Then, $\mathcal{H} = \mathcal{L}^2(S^1, \frac{dx}{2\pi}) \hat{\otimes} \mathbb{C}^2 = \mathcal{L}^2(S^1; \mathbb{C}^2)$ is the state space of the total system and

$$H = -\frac{d^2}{dx^2} \otimes \mathbb{I} + \mathbb{I} \otimes H_B \quad (45)$$

the total Hamiltonian. For this particular example we are going to change slightly the notation and will write the eigenstates corresponding to the Hermitian matrix H_B with the also usual notation for spin systems $|\uparrow\downarrow\rangle$. Therefore $H_B|\uparrow\downarrow\rangle = \pm\mu|\uparrow\downarrow\rangle$ and a particular element Φ of the composite system $\mathcal{H} = \mathcal{H}_A \hat{\otimes} \mathcal{H}_B$ will admit the decomposition $\Phi = \Phi^\uparrow \otimes |\uparrow\rangle + \Phi^\downarrow \otimes |\downarrow\rangle$. As boundary conditions we choose $U \in \mathcal{U}(\mathbb{C}_A^2) \otimes \mathcal{U}(\mathbb{C}_B^2)$ of the form $U = U_{A,\delta} \otimes U_B$, with $U_{A,\delta}$ as in eq. (44).

Physically, this system can be interpreted as follows (see Fig. 3). One has an charged particle rotating in a fixed orbit [As83]. The angular momentum of this particle depends on δ . In the center of this orbit, there is a fixed spin that interacts with a magnetic field of strength μ perpendicular to the plane of the orbit. The component U_B of the boundary condition shall be interpreted

as a macroscopic interaction *triggered* when the orbiting charged particle traverses an ideally infinitesimal region of the orbit.

We are going now to consider two different meaningful situations ⁴ for the boundary conditions corresponding to subsystem B . The first situation will correspond to select the unitary matrix U_B diagonal in the basis of H_B , namely:

$$U_B = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & -e^{i\alpha} \end{bmatrix} . \quad (46)$$

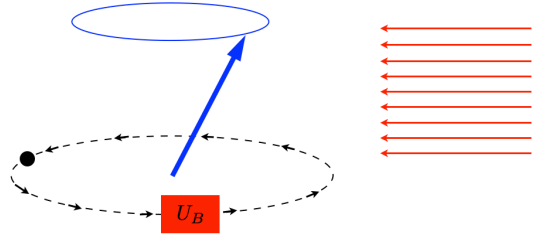


FIG. 3. Quantum Compass

Under this unitary matrices the boundary conditions defining the system take the explicit form:

$$\begin{aligned} \Phi^\uparrow(0) + i\Phi^{\uparrow'}(0) &= e^{i(\alpha+\delta)}(\Phi^\uparrow(1) + i\Phi^{\uparrow'}(1)) \\ \Phi^\uparrow(1) - i\Phi^{\uparrow'}(1) &= e^{i(\alpha-\delta)}(\Phi^\uparrow(0) - i\Phi^{\uparrow'}(0)) \\ \Phi^\downarrow(0) + i\Phi^{\downarrow'}(0) &= e^{i(-\alpha+\delta)}(\Phi^\downarrow(1) + i\Phi^{\downarrow'}(1)) \\ \Phi^\downarrow(1) - i\Phi^{\downarrow'}(1) &= e^{i(-\alpha-\delta)}(\Phi^\downarrow(0) - i\Phi^{\downarrow'}(0)) . \end{aligned}$$

One can proceed like in the previous examples and impose the above boundary conditions to the general solution of the spectral problem eq. (45) given by

$$\begin{aligned} \Phi^\uparrow(x) &= Ae^{i\sqrt{E-\mu}x} + Be^{-i\sqrt{E-\mu}x} \\ \Phi^\downarrow(x) &= Ce^{i\sqrt{E+\mu}x} + De^{-i\sqrt{E+\mu}x} , \end{aligned}$$

to find the corresponding spectral function associated to the problem, whose zeros are the eigenvalues. By doing so, one obtains the following spectral function:

⁴ Compare them with eq. (30).

$$\begin{aligned} \sigma_\alpha(E) = & \left[2i \sin(\sqrt{E-\mu}) + 2iE \sin(\sqrt{E-\mu}) 2i\mu \sin(\sqrt{E-\mu}) - 8\sqrt{E-\mu} \cos(\delta) e^{i\alpha} \dots \right. \\ & \dots + 8\sqrt{E-\mu} \cos(\sqrt{E-\mu}) \cos(\alpha) e^{i\alpha} - 2i(E-\mu+1) \sin(E-\mu) e^{i2\alpha} \left. \right] \times \left[2i \sin(\sqrt{E+\mu}) \dots \right. \\ & \dots + 2iE \sin(\sqrt{E+\mu}) + 2i\mu \sin(\sqrt{E+\mu}) - 8\sqrt{E+\mu} \cos(\delta) e^{-i\alpha} + 8\sqrt{E-\mu} \cos(\sqrt{E-\mu}) \cos(\alpha) e^{-i\alpha} \dots \\ & \left. \dots - 2i(E+\mu+1) \sin(E+\mu) e^{-i2\alpha} \right]. \end{aligned}$$

Finding the zeros of this transcendental function is a task that has to be done numerically. However, this task can be challenging, especially because σ_α is very close to vanish in some regions. Moreover, the information about the separability of the dynamical evolution depends on the eigenfunctions of the problem, that according to section IV need to admit a factorization $\Psi_l \otimes \rho_b$ where the indices l and b are independent of each other. It is worth to remark that it is not sufficient for guarantying separability that the eigenfunctions are separable. For all these reasons, in order to check that the above problem is not leading to separable dynamics, we will take the approach introduced in [Ib11]. There, an algorithm based on the Finite Element Method is introduced that is able to solve the spectral problem for any self-adjoint extension of a 1D Schrödinger problem. Then it is enough to use the isomorphism $\mathcal{L}^2([0,1]) \otimes \mathbb{C}^2 \simeq \mathcal{L}^2([0,1]) \oplus \mathcal{L}^2([0,1])$ to rewrite the problem given by eq. (45) into a form that can be handled by this numerical procedure. Figure 4 shows the eigenfunctions corresponding to the 6 smallest energies returned by the algorithm for $\mu = 10$, $\delta = \pi/2$, $\alpha = \pi/2$. The particular values of the energies are not shown because they are not relevant for the discussion. As it can be appreciated, the eigenfunctions are separa-

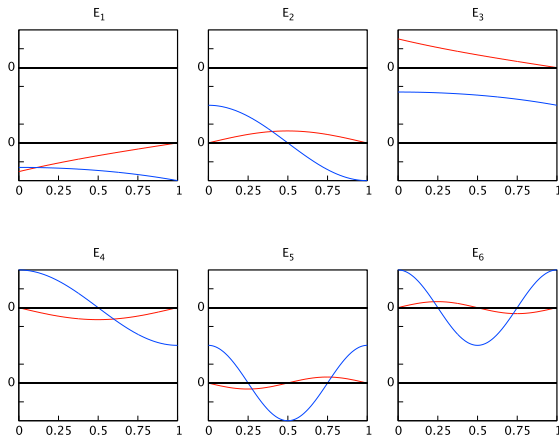


FIG. 4. Eigenfunctions of the 6 lowest energy levels for the case $\mu = 10$, $\delta = \pi/2$, $\alpha = \pi/2$. In each graph $\Phi^\uparrow(x)$ and $\Phi^\downarrow(x)$ are plotted simultaneously. Real parts are plotted in blue, imaginary parts in red.

ble states in this case. However, comparing the functions E_1 with E_3 , E_2 with E_4 and E_5 with E_6 , it can be seen that the imaginary parts (plotted in red) are changing sign between them. Hence we conclude that we have non-separable dynamics.

Now we consider a different situation where the unitary matrix U_B is taken antidiagonal with respect to the given basis of H_B and given by

$$U_B = \begin{bmatrix} 0 & e^{i\beta} \\ e^{-i\beta} & 0 \end{bmatrix}. \quad (47)$$

In this case the boundary conditions defining the system take the form:

$$\begin{aligned} \Phi^\uparrow(0) + i\Phi^{\uparrow'}(0) &= e^{i(\beta+\delta)} (\Phi^\downarrow(1) + i\Phi^{\downarrow'}(1)) \\ \Phi^\uparrow(1) - i\Phi^{\uparrow'}(1) &= e^{i(\beta-\delta)} (\Phi^\downarrow(0) - i\Phi^{\downarrow'}(0)) \\ \Phi^\downarrow(0) + i\Phi^{\downarrow'}(0) &= e^{i(-\beta+\delta)} (\Phi^\uparrow(1) + i\Phi^{\uparrow'}(1)) \\ \Phi^\downarrow(1) - i\Phi^{\downarrow'}(1) &= e^{i(-\beta-\delta)} (\Phi^\uparrow(0) - i\Phi^{\uparrow'}(0)). \end{aligned}$$

Again, one can compute the spectral function associated to this problem and we get:

$$\begin{aligned} \sigma_\beta(E) \propto & \sqrt{E^2 - \mu^2} \cos(\sqrt{E-\mu}) \cos(\sqrt{E+\mu}) - \dots \\ & \dots - E \sin(\sqrt{E-\mu}) \sin(\sqrt{E+\mu}) \dots \\ & \dots - \sqrt{E-\mu} \sqrt{E+\mu} \cos(2\delta). \end{aligned} \quad (48)$$

Surprisingly, the spectral function does not depend on the parameter β in this case, but the eigenfunctions do so. In Fig. 5 are plotted the eigenfunctions corresponding to the case $\mu = 10$, $\delta = \pi/2$, $\beta = \pi/2$. One can easily appreciate that they are non-separable.

VII. CONCLUSIONS AND DISCUSSION

Along the paper we have shown that it is possible by manipulating boundary conditions of a class of bipartite systems, transform a separable state into an entangled one. The reason for this phenomenon lies in the existence of many self-adjoint extensions of a bipartite symmetric system that lead to non-separable dynamics. We have been able to characterize all boundary conditions leading to separable dynamics in a class of symmetric bipartite systems and we have discussed a number of simple examples where this phenomenon is explicitly shown.

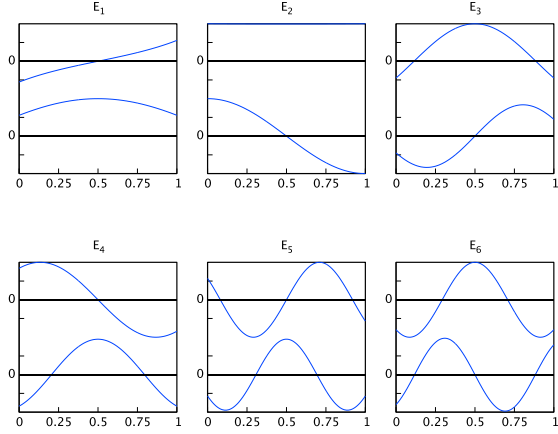


FIG. 5. Eigenfunctions of the 6 lowest energy levels for the case $\mu = 10$, $\delta = \pi/2$, $\beta = \pi/2$. In each graph $\Phi^\dagger(x)$ and $\Phi^\downarrow(x)$ are plotted simultaneously. Real parts are plotted in blue. The imaginary parts vanish identically in this case.

The systems exhibited are hybrid systems and one of them, the control or auxiliary system, must be symmetric but not self-adjoint or Hermitean. The most remarkable fact about this class of systems is that the space of self-adjoint extensions is much larger than the space of extensions of the control system and it incorporates boundary data that affect simultaneously the control and the controlled or bulk system. The controlled system above has a unitary dynamics, but together with the control system it becomes non-separable, hence taking the partial trace with respect to system A , will not give us back the original dynamics U_t^B . These ideas can be used to generate entangled states in a precise way, or to help to preserve entanglement without actually interacting with the “bulk” of the controlled system. The relation of these ideas with recent work on adiabatic computation and robust entanglement in hybrid systems will be pursued in the future.

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